

# On Hilbert's Tenth Problem

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## Abstract

Using an iterated Horner schema for evaluation of diophantine polynomials, we define a partial  $\mu$ -recursive “decision” algorithm *decis* as a “race” for a first *nullstelle* versus a first (internal) *proof* of non-nullity for such a polynomial – within a given theory  $\mathbf{T}$  extending Peano Arithmétique  $\mathbf{PA}$ . If  $\mathbf{T}$  is *diophantine sound*, i.e. if (internal) *provability* implies *truth* – for diophantine formulae –, then the  $\mathbf{T}$ -map *decis* gives *correct results* when applied to the codes of polynomial inequalities  $D(x_1, \dots, x_m) \neq 0$ . The additional hypothesis that  $\mathbf{T}$  be *diophantine complete* (in the syntactical sense) would guarantee in addition termination of *decis* on these formula, i.e. *decis* would constitute a *decision algorithm* for diophantine formulae in the sense of Hilbert’s 10th problem. From Matiyasevich’s impossibility for such a decision it follows, that a consistent theory  $\mathbf{T}$  extending  $\mathbf{PA}$  cannot be both diophantine sound and diophantine complete. We infer from this the existence of a *diophantine* formulae which is undecidable by  $\mathbf{T}$ . Diophantine correctness is inherited by the *diophantine completion*  $\tilde{\mathbf{T}}$  of  $\mathbf{T}$ , and within this extension *decis* terminates on all externally given diophantine polynomials, correctly. Matiyasevich’s theorem – for the strengthening  $\tilde{\mathbf{T}}$  of  $\mathbf{T}$  – then shows that  $\tilde{\mathbf{T}}$ , and hence  $\mathbf{T}$ , cannot be diophantine sound. But since the internal consistency formula  $\text{Con}_{\mathbf{T}}$  for  $\mathbf{T}$  implies – within  $\mathbf{PA}$  – diophantine soundness of  $\mathbf{T}$ , we get  $\mathbf{PA} \vdash \neg \text{Con}_{\mathbf{T}}$ , in particular  $\mathbf{PA}$  must derive its own internal inconsistency formula.

## Overview

- (i) Consider a theory  $\mathbf{T}$  with quantifiers and having terms for all primitive recursive maps (“p.r. maps”); so  $\mathbf{T}$  is to be PEANO Arithmétique  $\mathbf{PA}$  or one of  $\mathbf{PA}$ ’s extensions, e.g.  $\mathbf{ZF}$  or  $\mathbf{NGB}$ .

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- (ii) Obtain the theory  $\tilde{\mathbf{T}}$  by adding to  $\mathbf{T}$  the axiom  $\neg\text{Con}_{\mathbf{T}}$  of *internal inconsistency*. By GÖDEL's second incompleteness theorem,  $\tilde{\mathbf{T}}$  is consistent relative to  $\mathbf{T}$ .
- (iii)  $\mathbf{T}$  admits a  $\mu$ -recursive, partially defined “algorithm” *decis* aimed at deciding  $\mathbf{T}$ -internal (GÖDEL numbers of) p.r. predicates.
- (iv) By *internal semantical completeness* of  $\tilde{\mathbf{T}}$  with respect to p.r. predicates – involving *evaluation* of (GÖDEL numbers of) internal p.r. predicates – it is shown that in  $\tilde{\mathbf{T}}$  the *partial*  $\mu$ -recursive  $\mathbf{T}$ -map *decis* is in fact *total*, and that it gives *correct* results – the latter for arguments  $p$  of form  $p = \ulcorner \varphi \urcorner$ ,  $\varphi = \varphi(n)$  a p.r. predicate,  $\ulcorner \varphi \urcorner \in \mathbb{N}$  its internal GÖDEL number.
- (v) within  $\tilde{\mathbf{T}}$ , *decis* decides in particular (systems of) *diophantine equations*.
- (vi) MATIYASEVICH's negative result concerning this decision problem of HILBERT is a theorem of  $\mathbf{T}$ , a fortiori of  $\tilde{\mathbf{T}}$ .
- (vii) This contradiction shows  $\tilde{\mathbf{T}}$ , hence also  $\mathbf{T}$ , to be inconsistent: “unbounded formal quantification is incompatible with infinity.”

## 1 Decision

Crucial for the present approach to HILBERT's decision problem is availability – within  $\mathbf{T}$  – of a ( $\mu$ -recursive) *evaluation* map  $ev : \mathbb{N} \times \mathbb{N} \supset |\mathbb{N}, 2|_{\mathbf{PR}} \times \mathbb{N} \rightarrow 2$  on the  $\mathbf{T}$ -internal (primitive recursively decidable) set  $|\mathbb{N}, 2|_{\mathbf{PR}} \subset \mathbb{N}$  of GÖDEL numbers (“codes”) of p.r. predicates. (Primitive recursive *predicates* are viewed as p.r. *map terms* with codomain  $2 \subset \mathbb{N}$ ). This evaluation map  $ev$  is defined in  $\mathbf{T}$  by (*nested*) *double recursion* à la ACKERMANN, see PÉTER 1967, and satisfies the characteristic equation

$$ev(\ulcorner \varphi \urcorner, n) = \varphi(n)$$

for p.r. predicates  $\varphi = \varphi(n)$  of  $\mathbf{T}$ , cf. Appendix. Here  $\ulcorner \varphi \urcorner \in |\mathbb{N}, 2|_{\mathbf{PR}} \subset \mathbb{N}$  is  $\varphi$ 's  $\mathbf{T}$ -internal GÖDEL number.

Define now the partial  $\mu$ -recursive “decision”  $\tilde{\mathbf{T}}$ -map

$$decis = decis(p) : |\mathbb{N}, 2|_{\mathbf{PR}} \rightarrow 2$$

hoped for deciding (internal) p.r. predicates  $p$ , i.e.  $p \in |\mathbb{N}, 2|_{\mathbf{PR}} \subset \text{formulae}_{\tilde{\mathbf{T}}} = \text{formulae}_{\mathbf{T}} \subset \mathbb{N}$ , via the two “antagonistic” *termination indices*

$$\mu_{ex}(p), \mu_{thm}(p) : |\mathbb{N}, 2|_{\mathbf{PR}} \rightarrow \mathbb{N} \cup \{\infty\} \text{ as follows:}$$

$$\begin{aligned}\mu_{ex}(p) &:= \mu\{n : ev(p, n) = 0\} \quad \text{“minimal counterexample”} \\ &=_{def} \begin{cases} \min\{n : ev(p, n) = 0\} & \text{if } \exists n(ev(p, n) = 0) \\ \infty (\hat{=} \text{undefined}) & \text{if } \forall n(ev(p, n) = 1); \end{cases}\end{aligned}$$

the *theorem index*  $\mu_{thm}(p) \in \mathbb{N} \cup \{\infty\}$  of  $p \in |\mathbb{N}, 2|_{\mathbf{PR}}$  is defined by

$$\mu_{thm}(p) := \mu\{k : thm_{\tilde{\mathbf{T}}}(k) = p\};$$

here the p.r. enumeration  $thm_{\tilde{\mathbf{T}}} = thm_{\tilde{\mathbf{T}}}(k) : \mathbb{N} \rightarrow formulae_{\mathbf{T}} \subset \mathbb{N}$  is the  $\tilde{\mathbf{T}}$ -internal version of the *metamathematical* enumeration of all (GÖDEL numbers of)  $\tilde{\mathbf{T}}$ -theorems; enumeration is *lexicographic* by “length of shortest proof”.

Finally, we define the – a priori partial –  $\mu$ -recursive  $\mathbf{T}$ -map

$$\begin{aligned}decis &= decis(p) : |\mathbb{N}, 2|_{\mathbf{PR}} \rightarrow 2 \text{ by} \\ decis(p) &= \begin{cases} 0 & \text{if } \mu_{ex}(p) < \infty \quad (\text{“counterexample”}) \\ 1 & \text{if } \mu_{ex}(p) = \infty \text{ and } \mu_{thm}(p) < \infty \quad (\text{“theorem”}) \\ \infty & \text{otherwise, i. e. if } \mu_{thm}(p) = \mu_{ex}(p) = \infty. \end{cases}\end{aligned}$$

For proving *decis* to be totally defined within  $\tilde{\mathbf{T}} = \mathbf{T} + \neg\text{Con}_{\mathbf{T}}$  we rely on the following

**Lemma** (Internal Semantical Completeness):

$$\tilde{\mathbf{T}} \vdash \forall n(ev(p, n) = 1) \implies \exists k(thm_{\tilde{\mathbf{T}}}(k) = p)$$

with  $p$  free on  $|\mathbb{N}, 2|_{\mathbf{PR}}$ , in closed form:

$$\tilde{\mathbf{T}} \vdash (\forall p \in |\mathbb{N}, 2|_{\mathbf{PR}})[\forall n(ev(p, n) = 1) \implies \exists k(thm_{\tilde{\mathbf{T}}}(k) = p)].$$

**Proof:** One of the equivalent  $\mathbf{T}$ -formulae expressing internal inconsistency of  $\mathbf{T}$  is

$$\neg\text{Con}_{\mathbf{T}} = (\forall f \in formulae_{\mathbf{T}})(\exists k)(thm_{\mathbf{T}}(k) = f) :$$

“every internal formula (its GÖDEL number in  $\mathbf{T}$ ) is *provable*” (emphasis from GÖDEL). This gives in particular

$$\tilde{\mathbf{T}} \vdash \exists k(thm_{\tilde{\mathbf{T}}}(k) = p),$$

$p$  free on  $|\mathbb{N}, 2|_{\mathbf{PR}} \subset formulae_{\mathbf{T}} \subset \mathbb{N}$ , and hence – trivially – the assertion of the Lemma.

**Decision Lemma:**

- (i) within  $\tilde{\mathbf{T}} = \mathbf{T} + \neg \text{Con}_{\mathbf{T}}$ , the (a priori partial)  $\mu$ -recursive *decision-  
“algorithm”*

$$decis(p) : |\mathbb{N}, 2|_{\mathbf{PR}} \rightarrow 2$$

is in fact *totally defined*, with other words it *terminates* on all internal GÖDEL numbers  $p \in |\mathbb{N}, 2|_{\mathbf{PR}}$ .

- (ii) For  $\varphi = \varphi(n)$  a p.r. predicate,  $\ulcorner \varphi \urcorner \in |\mathbb{N}, 2|_{\mathbf{PR}} \subset \mathbb{N}$  its  $\mathbf{T}$ -internal GÖDEL number,  $decis(\ulcorner \varphi \urcorner)$  gives – in  $\tilde{\mathbf{T}}$  – the *correct* result:

$$\begin{aligned} - \tilde{\mathbf{T}} \vdash decis(\ulcorner \varphi \urcorner) = 0 &\iff \exists n(\neg \varphi(n)), \\ - \tilde{\mathbf{T}} \vdash decis(\ulcorner \varphi \urcorner) = 1 &\implies \forall n(\varphi(n)). \end{aligned}$$

**Proof** of (i):

$$\begin{aligned} \tilde{\mathbf{T}} \vdash [ \quad &\mu_{ex}(p) = \infty \\ &\iff \forall n(ev(p, n) = 1) \\ &\implies \exists k(thm_{\tilde{\mathbf{T}}}(k) = p) \\ &\quad \text{by internal semantical completeness of } \tilde{\mathbf{T}} \text{ above} \\ &\iff \mu_{thm}(p) < \infty \quad ]. \end{aligned}$$

Hence not both of  $\mu_{ex}(p), \mu_{thm}(p)$  can be undefined. This shows *termination*

$$decis(p) \in \{0, 1\}$$

of  $decis$  within  $\tilde{\mathbf{T}}$  for all (internal) p.r. predicates  $p$  (GÖDEL numbers thereof).

Proof of (ii):

$$\begin{aligned} \tilde{\mathbf{T}} \vdash [ \quad &decis(\ulcorner \varphi \urcorner) = 0 \\ &\iff \mu_{ex}(\ulcorner \varphi \urcorner) < \infty \\ &\iff \exists n(ev(\ulcorner \varphi \urcorner, n) = 0) \\ &\iff \exists n(\varphi(n) = 0) \quad \text{by } ev\text{'s evaluation property} \\ &\iff \exists n(\neg \varphi(n)) \quad ] \text{ as well as} \\ \tilde{\mathbf{T}} \vdash [ \quad &decis(\ulcorner \varphi \urcorner) = 1 \\ &\implies \mu_{ex}(\ulcorner \varphi \urcorner) = \infty \\ &\iff \forall n(ev(\ulcorner \varphi \urcorner, n) = 1) \\ &\iff \forall n(\varphi(n)) \quad ] \quad \text{q.e.d.} \end{aligned}$$

## 2 Hilbert's 10th Problem revisited

A system

$$D : \begin{array}{ccc} D_1^L(x_1, \dots, x_m) & = & D_1^R(x_1, \dots, x_m) \\ \vdots & & \vdots \\ D_k^L(x_1, \dots, x_m) & = & D_k^R(x_1, \dots, x_m) \end{array}$$

of  $k$  diophantine equations – see MATIYASEVICH 1993, 1.1, 1.2, and 1.3 – gives rise to a p.r. predicate

$$\begin{aligned} \varphi &= \varphi(x_1, \dots, x_m) : \mathbb{N}^m \rightarrow 2 \text{ defined by} \\ \varphi(x_1, \dots, x_m) &= [D_1^L \neq D_1^R \vee \dots \vee D_k^L \neq D_k^R] : \mathbb{N}^m \rightarrow 2 \end{aligned}$$

having the property that  $(x_1, \dots, x_m) \in \mathbb{N}^m$  is a solution to system  $(D)$  iff it is a *counterexample* to  $\varphi$ , and  $(D)$  has *no solution* (in natural numbers) iff  $\varphi$  holds for  $(x_1, \dots, x_m)$  free in  $\mathbb{N}^m$ .

CANTOR's p.r. enumeration  $cantor_m : \mathbb{N} \rightarrow \mathbb{N}^m$  having a p.r. inverse  $cantor_m^{-1} : \mathbb{N}^m \rightarrow \mathbb{N}$ ,

$$\psi = \psi(n) := \varphi(cantor_m(n)) : \mathbb{N} \rightarrow 2$$

is a p.r. predicate of  $\mathbf{T}$  such that  $(x_1, \dots, x_m) \in \mathbb{N}^m$  solves  $(D)$  iff  $cantor_m^{-1}(x_1, \dots, x_m) \in \mathbb{N}$  is a *counterexample* to  $\psi$ , and  $(D)$  is *unsolvable* iff  $\psi(n)$  holds for  $n$  free in  $\mathbb{N}$ . So from the Decision Lemma (for p.r. predicates) above we obtain:

### Decision Theorem:

- (i) Within the – somewhat strange – theory  $\tilde{\mathbf{T}} = \mathbf{T} + \neg\text{Con}_T$ , the (partial)  $\mu$ -recursive map (the “algorithm”)  $decis : |\mathbb{N}, 2|_{\mathbf{PR}} \rightarrow 2$  decides all (internal) primitive recursive predicates, in particular all (internal, a fortiori external) GÖDEL numbers coding “diophantine” predicates as considered above, and hence decides internal, a fortiori external (systems of) *Diophantine equations*.
- (ii) Since  $\mu$ -recursion and TURING-machines have equal *computation power* – by the verified part of CHURCH's thesis – this means: Within  $\tilde{\mathbf{T}}$ ,  $decis$  gives rise to a TURING machine  $TM$  deciding all internally given as well as all externally given Diophantine equations, i.e.  $\tilde{\mathbf{T}}$  admits a *positive* solution to HILBERT's 10th problem.
- (iii) On the other hand, MATIYASEVICH's *negative* solution to this problem needs as a formal framework  $\mathbf{T}$  just Arithmétique  $+\exists$ .

- (iv) The latter two results – MATIYASEVICH’s *negative  $\mathbf{T}$ -theorem* and our *positive  $\tilde{\mathbf{T}}$ -theorem* contradict each other in the stronger theory  $\tilde{\mathbf{T}}$ . This shows  $\tilde{\mathbf{T}}$  to be *inconsistent*.
- (v) GÖDEL’s consistency of  $\neg\text{Con}_{\mathbf{T}}$  relative to  $\mathbf{T}$  then entails inconsistency of  $\mathbf{T}$ , whence in particular inconsistency of PEANO Arithmétique  $\mathbf{PA}$  and of the classical set theories.

**Corollary:** Since MATIYASEVICH 1993 makes essential use of formal (existential) quantification for “unsolving” HILBERT’s 10th problem, this only decision problem on HILBERT’s list is again open – for treatment within the framework of a suitable *constructive* foundation for Arithmetic.

### 3 Appendix: Evaluation

In section 2 we made appeal to availability in  $\mathbf{T}$  of an *evaluation*  $ev = ev(p, n)$  of (internal) p.r. predicate codes  $p$  satisfying

$$ev(\ulcorner \varphi \urcorner, n) = \varphi(n)$$

for (“external”) p.r. predicates  $\varphi : \mathbb{N} \rightarrow 2$  in  $\mathbf{T}$ . We identify a p.r. predicate  $\varphi = \varphi(n)$  of  $\mathbf{T}$  with its associated p.r. map term  $\varphi = \varphi(n) : \mathbb{N} \rightarrow 2$ , since we want to define the evaluation of (internal) p.r. predicates by restriction of an evaluation of *all* internal p.r. map terms out of the set  $|\mathbb{N}, 2|_{\mathbf{PR}} \subset \mathbb{N}$  of (internal) p.r. map terms from  $\mathbb{N}$  to 2.

For *defining* this map term evaluation  $ev$  by (*nested*) *double recursion* à la ACKERMANN (cf. PÉTER 1967) we need a *universal set* (object)

$$\mathbb{U} = \mathbb{N}^{(*)}$$

of all *nested pairs* of natural numbers, and hence containing all  $\mathbf{PR}$ -objects  $1, \mathbb{N}, \dots, A, \dots, B, A \times B, \dots$  as *disjoint* (exception:  $1 \subset \mathbb{N}$ ) p.r. decidable subsets.

This set  $\mathbb{N}^{(*)}$  is directly available in *set theory*. Within PEANO Arithmétique, it can be “constructed” via *coding* as a decidable subset of  $\mathbb{N}$ .

**Definition:** *Evaluation*

$$ev = ev(u, a) : \mathbb{N} \times \mathbb{N}^{(*)} \supset PR \times \mathbb{N}^{(*)} \rightarrow \mathbb{N}^{(*)}$$

of the *internal* (GÖDEL numbers of) p.r. maps  $u, v, w \in PR \subset \mathbb{N}$ , on binary nested tupels  $a, b, c \in \mathbb{N}^{(*)}$  of natural numbers is now defined by (*nested*) *double recursion* with principal recursion parameter “operator-depth”  $depth(u)$  of  $u$  as follows:

- basic internal map terms  $\ulcorner 0 \urcorner, \ulcorner s \urcorner, \ulcorner id \urcorner, \ulcorner ! \urcorner, \ulcorner \Delta \urcorner, \ulcorner \Theta \urcorner, \ulcorner \ell \urcorner$  :
  - $ev(\ulcorner 0 \urcorner, 0) = 0 = 0(0) \in \mathbb{N}$  “zero map”,
  - $ev(\ulcorner s \urcorner, n) = n + 1 = s(n) \in \mathbb{N}$  “successor map”,
  - $ev(\ulcorner id \urcorner, a) = a = id(a)$  “identity”,
  - $ev(\ulcorner ! \urcorner, a) = 0 = !(a) \in 1 \subset \mathbb{N}$  “terminal map”,
  - $ev(\ulcorner \Delta \urcorner, a) = (a, a) = \Delta(a)$  “diagonal”,
  - $ev(\ulcorner \Theta \urcorner, (a, b)) = (b, a) = \Theta(a, b)$  “transposition”,
  - $ev(\ulcorner \ell \urcorner, (a, b)) = a = \ell(a, b)$  “left projection”.

This defines  $ev$  on **PR**’s (map-)constants, *depth* of these “basic” map terms is set to 1.

We now define  $ev$  on compound internal p.r. map terms:

- internally *composed*  $v \ulcorner \circ \urcorner u$ :

$$ev(v \ulcorner \circ \urcorner u, a) = ev(v, ev(u, a)).$$

This definition is *legitimate*, since

$$\begin{aligned} depth(u), depth(v) &< depth(v \ulcorner \circ \urcorner u) \\ &=_{def} \max(depth(u), depth(v)) + 1 \in \mathbb{N}; \end{aligned}$$

*Example:*

$$\begin{aligned} &ev(\ulcorner s \urcorner \ulcorner \circ \urcorner \ulcorner s \urcorner \ulcorner \circ \urcorner \ulcorner s \urcorner, s(0)) \\ &= ev(\ulcorner s \urcorner, ev(\ulcorner s \urcorner, ev(\ulcorner s \urcorner, s(0)))) \\ &= ((s(0) + 1) + 1) + 1 = 4. \end{aligned}$$

- cylindrified  $\ulcorner id \urcorner \ulcorner \times \urcorner v$  :

$$ev(\ulcorner id \urcorner \ulcorner \times \urcorner v, (a, b)) = (a, ev(v, b)),$$

“evaluation in the second component”.

*legitimacy of this definition:*

$$depth(v) < depth(\ulcorner id \urcorner \ulcorner \times \urcorner v) =_{def} depth(v) + 1.$$

- internally *iterated*  $u^{\S}$ :

$$\begin{aligned} ev(u^{\S}, (a, 0)) &= a, \\ ev(u^{\S}, (a, n + 1)) &= ev(u, ev(u^{\S}, (a, n))). \end{aligned}$$

This last case is in fact a (*nested*) *double recursion* à la ACKERMANN, since the *internally iterated*  $u^{\S}$  of  $u$  is evaluated in a p.r. manner with respect to the second parameter  $n \in \mathbb{N}$  – which is to count

the iteration loops still to be performed. The principal recursion parameter is (internal) operator-depth  $depth = depth(u) : \mathbb{N} \supset PR \rightarrow \mathbb{N}$ , in particular in this last case  $depth(u^{\S}) =_{def} depth(u) + 1$ .

Each primitive recursive map can be generated from the basic maps  $0, s, id, !, \Delta, \Theta$ , and  $\ell$  by composition, cylindrification and iteration: *substitution* is realized via composition with the induced  $(f, g) = (f, g)(c) = (f(c), g(c))$  which in turn is obtained via diagonal, cylindrification, transposition, and composition. Since iteration  $g^{\S}$  then gives the (“full”) schema of primitive recursion (see FREYD 1972, PFENDER et al. 1994), *ev* in fact evaluates all GÖDEL numbers of (internal) p.r. map terms, recursively given in the above way.

Let us call  $\mathbf{PR} + ev$  the extension of  $\mathbf{PR}$  by a (formal) map

$$ev = ev(u, a) : \mathbb{N} \times \mathbb{N}^{(*)} \supset PR \times \mathbb{N}^{(*)} \rightarrow \mathbb{N}^{(*)}$$

satisfying the above 2-recursive system for *ev*.

For our “set” theory  $\mathbf{T}$  we now prove the following

**Evaluation Lemma:** For primitive recursive  $f : \mathbb{N}^{(*)} \supset A \rightarrow B \subset \mathbb{N}^{(*)}$  in  $\mathbf{T}$ ,  $\mathbf{T}$  extending  $\mathbf{PR} + ev$ , we have

$$ev(\ulcorner f \urcorner, a) = f(a) : A \rightarrow B,$$

in particular for  $\varphi : \mathbb{N} \rightarrow 2$  (the map term representing) a p.r. predicate of  $\mathbf{T}$  :

$$ev(\ulcorner \varphi \urcorner, n) = \varphi(n) : \mathbb{N} \rightarrow 2, \quad n \text{ free variable on } \mathbb{N}.$$

**Proof** by *external* (“metamathematical”) *induction* on the operator-depth  $\mathbf{depth}(f) \in \mathbf{N}$  of  $f$  varying on  $\mathbf{PR} \subset \mathbf{N}$ , in case of an iterated  $f = g^{\S}(a, n) : A \times \mathbb{N} \rightarrow A$  this external induction will be combined with an internal induction on the *iteration parameter*  $n \in \mathbb{N}$ .  $\mathbf{depth} : \mathbf{PR} \rightarrow \mathbf{N}$  is the external primitive recursive “twin” of  $depth : PR \rightarrow \mathbb{N}$  above; it is characterised by  $depth(\ulcorner f \urcorner) = num(\mathbf{depth}(f))$  for  $f : A \rightarrow B$  in  $\mathbf{PR} \subset \mathbf{T}$ . Here  $num = num(\mathbf{n}) : \mathbf{N} \rightarrow \mathbf{T}(1, \mathbb{N})$  maps each *external* natural number  $\mathbf{n}$  into its corresponding **T-numeral**, as defined e.g. in set theory by associating VON NEUMANN numerals.

- Anchoring: the assertion holds for the *basic* maps  $0, \dots, \ell$  (with  $\mathbf{depth}$  set to  $\mathbf{1} \in \mathbf{N}$ ) just by definition of *ev*.



- composition case  $f = h \circ g : A \rightarrow B \rightarrow C$  :

$$\begin{aligned}
ev(\ulcorner f \urcorner, a) &= ev(\ulcorner h \circ g \urcorner, a) \\
&= ev(\ulcorner h \urcorner \ulcorner \circ \urcorner \ulcorner g \urcorner, a) \quad \text{since } \ulcorner \circ \urcorner \text{ internalizes 'o'} \\
&= ev(\ulcorner h \urcorner, ev(\ulcorner g \urcorner, a)) \quad \text{by definition of } ev \\
&= ev(\ulcorner h \urcorner, g(a)) \quad \text{by recursion hypothesis on } g \\
&\quad \text{since } \mathbf{depth}(g) < \mathbf{depth}(f) \\
&= h(g(a)) \quad \text{by recursion hypothesis on } h \\
&\quad \text{since } \mathbf{depth}(h) < \mathbf{depth}(f) \\
&= (h \circ g)(a) = f(a).
\end{aligned}$$

- case  $f = id \times g : A \times B \rightarrow A \times C$  a cylindrified map:

$$\begin{aligned}
ev(\ulcorner f \urcorner, (a, b)) &= ev(\ulcorner id \times g \urcorner, (a, b)) \\
&= ev(\ulcorner id \urcorner \ulcorner \times \urcorner \ulcorner g \urcorner, (a, b)) \\
&\quad \text{since } \ulcorner \times \urcorner \text{ is to internalize } \times \\
&= (a, ev(\ulcorner g \urcorner, b)) \quad \text{by definition of } ev \\
&= (a, g(b)) \quad \text{by recursion hypothesis on } g \\
&\quad \text{since } \mathbf{depth}(g) < \mathbf{depth}(f) \\
&= (id \times g)(a, b) = f(a, b).
\end{aligned}$$

- The remaining case – not quite so simple – is that of an *iterated*  $f = g^{\S} : A \times \mathbb{N} \rightarrow A$  of a (p.r.) endo map  $g : A \rightarrow A$ ,  $g^{\S}$  characterized by

$$g^{\S}(a, 0) = a, \quad g^{\S}(a, n+1) = g(g^{\S}(a, n)) :$$

the assertion of the Lemma holds in this last case too, since – “anchoring”  $n = 0$  for internal induction:

$$\begin{aligned}
ev(\ulcorner f \urcorner, (a, 0)) &= ev(\ulcorner g^{\S} \urcorner, (a, 0)) \\
&= ev(\ulcorner g \urcorner^{\S}, (a, 0)) = a \quad \text{since } \langle \_ \rangle^{\S} \text{ internalizes } (-)^{\wedge} \\
&= g^{\S}(a, 0) = f(a, 0)
\end{aligned}$$

- as well as (internal induction step, using the external recursion

hypothesis):

$$\begin{aligned}
ev(\ulcorner f \urcorner, (a, n+1)) &= ev(\ulcorner g^{\S} \urcorner, (a, n+1)) \\
&= ev(\ulcorner g^{\S} \urcorner, (a, n+1)) \quad \text{since } \langle \_ \rangle^{\S} \text{ internalizes } (\_)^\wedge \\
&= ev(\ulcorner g^{\S} \urcorner, ev(\ulcorner g^{\S} \urcorner, (a, n))) \\
&\quad \text{by (internal) inductive definition of } ev \\
&\quad \text{in the present case } v = u^{\S} = \ulcorner g^{\S} \urcorner \\
&= ev(\ulcorner g^{\S} \urcorner, ev(\ulcorner g^{\S} \urcorner, (a, n))) \quad \text{by } \langle \_ \rangle^{\S} \text{ internalizing } (\_)^\wedge \\
&= ev(\ulcorner g^{\S} \urcorner, g^{\S}(a, n)) \quad \text{by (internal) induction hypothesis on } n \\
&= g^{\S}(a, n) \quad \text{by (external) recursion hypothesis on } g \\
&\quad \text{since } \mathbf{depth}(g) < \mathbf{depth}(f) \\
&= g^{\S}(a, n+1) = f(a, n+1) \text{ by definition of the iterated } g^{\S} \text{ q.e.d.}
\end{aligned}$$

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